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The Webster method of apportionment

(allocation/Congress/representation/fair division/U.S. Constitution)

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ABSTRACT **Several results concerning the problem of U.S. Congressional apportionment are given which, together, indicate that a method first proposed by Daniel Webster (also known as the "Major Fractions" method) seems fairest when judged on the basis of criteria suggested by common sense and precedent.**

The Constitution of the United States requires that the House of Representatives be apportioned among the several states according to their populations. Various methods for doing this have been advanced over the years, beginning in 1792 after the first census. Four different methods have been used. From study of the differences between the methods, there emerge several criteria which we believe to be most important by reason of common sense, Constitutional requirement, and precedent.

There exists no method that satisfies all desirable criteria. One essential property, population monotonicity, rules out the Quota method (1, 2) and, indeed, any method satisfying quota (3, 4). The aim of this note is to set down, for the record, several results describing the interplay between these criteria which, together, indicate that one method—that of Webster—best answers the needs. Detailed proofs will appear elsewhere.

Definitions and elementary properties

An *apportionment problem* is specified by an s vector ($s \geq 2$) of rational numbers $\mathbf{p} = (p_1, \ldots, p_s)$, all $p_i > 0$, and an integer house size $h \geq 0$. An *apportionment of h among s* is an integer s vector $\mathbf{a} = (a_1, \ldots, a_s) \geq 0$ with $\Sigma_i a_i = h$. An *apportionment method* is a multivalued function $\mathbf{M}(\mathbf{p};h)$ such that, for each $\mathbf{p} > 0$ and $h \geq 0$, $\mathbf{M}$ is a set of apportionments $\mathbf{a}$ of h among s (sometimes unique, sometimes not). A *particular* $\mathbf{M}$-*solution* is a single-valued function $\mathbf{f}$, with $\mathbf{f}(\mathbf{p};h) = \mathbf{a} \in \mathbf{M}(\mathbf{p},h)$.

The *quota* of state i for h,s is $q_i = p_i h/\Sigma_j p_j$. The *lower quota* is $\lfloor q_i \rfloor$; the *upper quota* is $\lceil q_i \rceil$.[§]

The following elementary properties define more explicitly what is meant by a method that apportions "according to numbers." Method $\mathbf{M}$ is *homogeneous* when $\mathbf{a} \in \mathbf{M}(\lambda\mathbf{p};h)$ if and only if $\mathbf{a} \in \mathbf{M}(\mathbf{p};h)$ for all rational $\lambda > 0$. A method is *symmetric* if, for any permutation π of $1, \ldots, s$, $(a_{\pi(1)}, \ldots, a_{\pi(s)}) \in \mathbf{M}[(p_{\pi(1)}, \ldots, p_{\pi(s)}); h]$ if and only if $\mathbf{a} \in \mathbf{M}(\mathbf{p};h)$. Thus, only the numbers count, not the names of states.

For any s vectors $\mathbf{x},\mathbf{y} \geqq 0$, $\mathbf{x}$ is proportional to $\mathbf{y}$, written $\mathbf{x} \propto \mathbf{y}$, if $\mathbf{x} = \alpha\,\mathbf{y}$ for some $\alpha > 0$. A method $\mathbf{M}$ is *proportional* if for any populations $\mathbf{p}$ and integer $\mathbf{a} > 0$, $\mathbf{a} \propto \mathbf{p}$ implies $\mathbf{M}(\mathbf{p},\Sigma a_i) = \{\mathbf{a}\}$; and $\mathbf{b} \propto \mathbf{c} \in \mathbf{M}(\mathbf{p},\Sigma c_i), \Sigma b_i \leq \Sigma c_i$ implies $\mathbf{b} \in \mathbf{M}(\mathbf{p},\Sigma b_i)$. These conditions are essential to the very idea of proportionality.

Finally, a method is *complete* if $\mathbf{p}^n \rightarrow \mathbf{p}$ and $\mathbf{a} \in \mathbf{M}(\mathbf{p}^n;h)$ for all n implies $\mathbf{a} \in \mathbf{M}(\mathbf{p};h)$. So, if the $\mathbf{p}^n$ are a sequence of increasingly accurate estimates of the true population $\mathbf{p}$, all of which admit the apportionment $\mathbf{a}$ by $\mathbf{M}$, then so does $\mathbf{p}$. This is a technical property that allows for a just handling of ties.

These four properties are met by all methods which have, to our knowledge, ever been proposed, and we assume them in the sequel unless otherwise noted.

Divisor methods

A *rank-index* $r(p,a)$, $a \geqq 0$ integer and $p > 0$ rational is any real valued function satisfying $r(p,a) > r(p,a+1) \geq 0$. $r(p,a)$ may be plus infinity. The *Huntington method based on* $r(p,a)$ is

$$\mathbf{M}(\mathbf{p};h) = \{\mathbf{a} \geqq 0{:}a_i \text{ integer}, \Sigma a_i = h, \max_i r(p_i,a_i) \leq \min_{a_j>0} r(p_j,a_j - 1)\}.$$

A rank-index determines a method by assigning priorities in the allocation of seats by the following recursive rule on the size of the house ($h' \leq h$): at $h' = 0$ set all $a_i = 0$; if $\mathbf{a}$ apportions $h' < h$, then an apportionment of $h' + 1$ seats is found by giving one more seat to some state maximizing $r(p_i,a_i)$.

A *divisor criterion* $d(a)$, $a \geqq 0$ integer, is any non-negative monotone increasing function. The *divisor method based on* $d(a)$ is the Huntington method based on $r(p,a) = p/d(a)$. We adopt the convention that $p > q$ implies $p/0 > q/0$.

By virtue of the proportionality assumption we have the following.

LEMMA 1. *Every divisor method can be represented by a divisor criterion satisfying*

(i) $a \leq d(a) \leq a + 1$ *for all integer* $a \geqq 0$; *and*
(ii) $a_1/b_1 = a_2/b_2 > 1$ *implies*
$d(b_1)/d(b_2 - 1) \geq d(a_1)/d(a_2 - 1) > d(a_1 - 1)/d(a_2) \geqq d(b_1 - 1)/d(b_2)$
for all integers $a_1, a_2, b_1, b_2 > 0$.

It is of interest to know that virtually all of the methods proposed—with the notable exception of Hamilton's—have been such divisor methods. These have received different names and descriptions in various countries and times. To the best of our knowledge they should be credited in terms of earliest discovery as follows. The method of John Quincy Adams (5) has $d(a) = a$. The method of James Dean (6), Professor of Astronomy and Mathematics at Dartmouth and the University of Vermont, has $d(a) = 2a(a + 1)/(2a + 1)$. The "equal proportions" method of E. V. Huntington (7, 8), Professor of Mathematics at Harvard, has $d(a) = \sqrt{a(a + 1)}$. The method of Daniel Webster (6) has $d(a) = a + 1/2$. The method of Thomas Jefferson (9) has $d(a) = a + 1$.

Huntington unified these "historic five methods" through his test of inequality approach (7, 8) and showed how they could be computed recursively by using divisor criteria. In the 18th and 19th centuries the methods were described in different

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[§] $\lfloor x \rfloor$ denotes the greatest integer $\leq x$; $\lceil x \rceil$ denotes the smallest integer $\geq x$.

(although equivalent) terms using the idea of an *ideal district size* or *common divisor*, λ. First a λ is specified; then, the numbers p_i/λ are used to determine the apportionments a_i, whose sum determines h. For example, Adams' method rounds up all fractions (that is, sets $a_i = \lceil p_i/\lambda \rceil$); Jefferson's method drops all fractions (that is, sets $a_i = \lfloor p_i/\lambda \rfloor$); and Webster's method rounds to the nearest integer (that is, sets $a_i = [p_i/\lambda + \frac{1}{2}]$). Any divisor method may be described in similar terms by using a divisor λ.

Jefferson's method was used for the apportionments based on the censuses of 1790 through 1830. Webster's method was used for 1840, 1900, 1910, and 1930. Huntington's method agreed with Webster's in 1930, and since 1940 it has been the law of the land.

House monotonicity

Another early method is Alexander Hamilton's (10), reinvented and used for the censuses of 1850, 1880, and 1890 under the name "Vinton's Method of 1850" (although in each instance the apportionment agreed with Webster's).¶ It gives to each state i its lower quota $\lfloor q_i \rfloor$ and then assigns one additional seat to each of the $\Sigma(q_i - \lfloor q_i \rfloor)$ states having the largest remainders $q_i - \lfloor q_i \rfloor$. But it admits the infamous Alabama paradox: an increase in the house can result in a state losing seats.

A method **M** is *house monotone* if there exists for any **p** some **M** solution f for which $\mathbf{f}(\mathbf{p};h+1) \geqq \mathbf{f}(\mathbf{p};h)$ for all h. Congressional debate makes clear that only house monotone methods can be countenanced. All Huntington methods are house monotone; indeed, the quest for house monotone methods is what motivated the work of Willcox (11, 12) and then of Huntington (7, 8).

Uniformity

An inherent principle of fair division is: every part of a fair division must be fair. In the context of apportionment, this principle can be formulated as follows: **M** is *uniform* (13) if $(\mathbf{a},\mathbf{b}) \in \mathbf{M}(\mathbf{p},\mathbf{q};h)$ implies (*i*) $\mathbf{a} \in \mathbf{M}(\mathbf{p};\Sigma_i a_i)$, and (*ii*) if also $\mathbf{a}' \in \mathbf{M}(\mathbf{p};\Sigma a_i)$ then $(\mathbf{a}',\mathbf{b}) \in \mathbf{M}(\mathbf{p},\mathbf{q};h)$. That is, an apportionment acceptable for all states is acceptable if restricted to any subset of states considered alone; moreover, if the restriction admits a different apportionment of the same number of seats, then using it instead results in an alternate acceptable apportionment for the whole.

THEOREM 1. *If a method is uniform, then it is house monotone.*

In fact the theorem holds for nonproportional methods; it is only necessary to require that two states having identical populations cannot have apportionments differing by more than one seat. [This result was later independently noted by Hylland (14).] Because the Hamilton method is not house monotone, it is not uniform.

THEOREM 2. *A method is uniform if and only if it is a Huntington method.*

This follows directly from an earlier characterization of Huntington methods (15) and *Theorem 1.*

Fixing the house size and number of states

Uniformity inherently bears the idea that a method should be applicable to all problems with all possible house sizes and numbers of states. A critic might counter that, in many situations, s and h are fixed: in the United States, $h = 435$ and $s = 50$. So, let us fix s and h.

An (s,h)*-method* $\mathbf{M}^*(\mathbf{p})$ is a method $\mathbf{M}(\mathbf{p},h')$ restricted to s vectors **p** and $h' = h$. A *divisor* (s,h)*-method* is an (s,h)-method $\mathbf{M}^*(\mathbf{p})$ for which there is a divisor criterion $d(a)$, depending on s and h, such that $(a_1, \ldots, a_s) = \mathbf{a} \in \mathbf{M}^*(\mathbf{p})$ if and only if $\mathbf{a} \geqq 0$, $\Sigma a_i = h$, and $\min_{a_i>0} p_i/d(a_i - 1) \geqq \max_j p_j/d(a_j)$. Even though a divisor (s,h)-method is proportional, it may not be representable by a divisor criterion $d(a)$ satisfying the conditions of *Lemma 1.*

Population monotonicity

A census provides populations $\mathbf{p} = (p_1, \ldots, p_s)$. But these change over time, and errors in census numbers may yield various **p**s. A method must behave reasonably when applied to different **p**s. Many definitions for such behavior are conceivable. The obvious mathematical choice is to compare two **p**s identical in all state populations save one, and ask that a method never assign to the one state having greater population fewer seats. Actual population changes over the years do not tend to produce such situations.

An (s,h)-method $\mathbf{M}^*(\mathbf{p})$ is *population monotone* if $\mathbf{a} \in \mathbf{M}^*(\mathbf{p})$, $\mathbf{a}' \in \mathbf{M}^*(\mathbf{p}')$ and $p_i'/p_j' \geqq p_i/p_j$ imply that $a_i' < a_i$ and $a_j' > a_j$ occurs only if $p_i'/p_j' = p_i/p_j$ and $(a_1, \ldots, a_i', \ldots, a_j', \ldots, a_s) \in \mathbf{M}^*(\mathbf{p})$. This avers that, if populations change, apportionments should not change by giving more seats to a state with relatively smaller population and less seats to a state with relatively greater population (unless there is a "tie").

THEOREM 3. *Fix* $s \neq 3$ *and* $h \geqq s$. *An* (s,h)*-method is population monotone if and only if it is a divisor* (s,h)*-method.*

The result is not true for $s = 3$: a counterexample exists for $h = 7$.

COROLLARY. *A method* **M** *is uniform and population monotone if and only if it is a divisor method.*

Invoking uniformity together with population monotonicity results in a divisor independent of s and h, which is what one would naturally expect. In fact, we have shown that uniformity, together with the very weak demand that $p_i > p_j$ must imply $a_i \geqq a_j$, suffices to characterize divisor methods. A somewhat more general result obtains if proportionality is dropped (16). [Hylland (14) has recently found a similar result.]

Satisfying quota

The most primitive request of a method of apportionment is that it should guarantee to each state at least its lower quota and at most its upper quota, $\lfloor q_i \rfloor \leqq a_i \leqq \lceil q_i \rceil$, for all i. Methods with this property are said to *satisfy quota*. The Hamilton method is predicated on it, as is the Quota method (1, 2). It is an unfortunate fact that it is simply impossible to have a method that satisfies quota together with other fundamental criteria.

THEOREM 4. *There is no uniform method that satisfies quota.*

THEOREM 5. *Fix* $s \geqq 4$ *and* h *large* ($h \geqq s+3$ *suffices*). *There is no population monotone* (s,h)*-method that satisfies quota.*

So even for fixed s there is no method which reconciles the primitive wish to satisfy quota with the necessity of population monotonicity. For $s = 3$ a special result obtains.

THEOREM 6. *The method of Webster is the unique divisor method which satisfies quota for* $s = 3$.

Satisfying quota—as desirable as it may be—is incompatible with uniformity and with population monotonicity for fixed s and h. We conclude that it must be abandoned. And this, we will see, can be done at essentially no cost. In particular, we discard the Quota method (1, 2) as well as all quotatone methods (3, 4).

We are left with the class of divisor methods.

¶ For the censuses of 1860 and 1870, Congress brought *ad hoc* changes to initially propose Hamilton apportionments.

Bias

Why has Huntington's method been retained for U.S. Congressional apportionment from among the five historic divisor methods? If one inspects examples, it is immediately evident that, as application of Adams' method is succeeded by application of Dean's, and then Huntington's, Webster's and Jefferson's, solutions tend more and more to favor large states over small. This behavior can be proved (2). Two reasons were advanced in reports to the National Academy of Sciences (17, 18) to adopt Huntington's method: (*i*) it is in the middle of the five from the point of view of favoring small versus large,‖ (*ii*) it is based on a measure of pairwise inequality of representation between states which (while arbitrary) seems preferable to those measures of inequality which characterize the other four methods. In these reports no absolute standard for determining whether a method favors small or large states was set down.

The desire to choose a method which is "unbiased" in its award of seats to small and large states is well founded and is rooted in the "historic compromise" of the 1787 Constitutional Convention by which the Senate was given representation independent of population and each state was assured of at least one seat in the House. Moreover, bias can be given a definite meaning.

An apportionment that gives a_1 and a_2 seats to states having populations $p_1 > p_2 > 0$ *favors the larger state over the smaller state* if $a_1/p_1 > a_2/p_2$, and *favors the smaller state over the larger state* if $a_1/p_1 < a_2/p_2$. It may be asked whether, over many pairs (p_1,p_2), $p_1 > p_2$, a method tends more often to favor the larger state over the smaller or vice versa.

A simple test is based on the choice of any pair of populations (p_1,p_2), $p_1 > p_2 > 0$. Two states having these populations could divide any number of seats h between them. Because the methods we are considering are uniform, the way two states share h seats is determined without reference to any other states in the problem. Consider all possible house sizes h shared between two states with populations (p_1,p_2) up to the *perfect house* $h^*(p_1,p_2)$, defined to be the least h such that both states have integer quotas. For a given method **M**, let $\sigma(p_1,p_2)$ be the number of pairs (a_1,a_2), $0 < a_1 + a_2 = h < h^*(p_1,p_2)$, with (a_1,a_2) an **M**-apportionment for (p_1,p_2) and h, such that state 2 (the smaller state) is favored over state 1 (the larger state). Similarly, define $\tau(p_1,p_2)$ to be the number of pairs (a_1,a_2) that favor state 1 over state 2.

M is *pairwise unbiased on populations* if for every $p_1 > p_2 > 0$, $\sigma(p_1,p_2) = \tau(p_1,p_2)$.

THEOREM 7. *Webster's is the unique divisor method that is pairwise unbiased on populations.*

The foregoing approach to bias has the merit that no assumption is made about the distribution of populations. However, its use as an empirical test is limited because actual apportionments typically involve numbers of seats much smaller than $h^*(p_1,p_2)$.

Instead of fixing $p_1 > p_2 > 0$, one may fix integers $a_1 > a_2 > 0$ and ask how often, over all populations pairs (p_1,p_2) such that $(a_1,a_2) \in \mathbf{M}(p_1,p_2;a_1 + a_2)$, state 1 is favored over state 2. Note that the population monotonicity of **M** implies $p_1 \geq p_2$.

Assume as a theoretical model that all (rational) populations $(p_1,p_2) = \mathbf{p}$ are equally likely in any bounded measurable set $X \subset \mathbf{R}^2_+$; thus, for any measurable $Y \subset X$ the probability that $\mathbf{p} \in Y$, given $\mathbf{p} \in X$, is the Lebesgue measure of Y in X. For any divisor $\lambda > 0$ and any pair of integers $\mathbf{a} = (a_1,a_2)$, $a_1 > a_2 > 0$, define the sample space $X^\lambda(\mathbf{a})$ to be the set of all $\mathbf{p}$ which yield the apportionment $\mathbf{a}$ when divisor method **M**, having criterion $d(a)$, is used with the given divisor λ. Thus $X^\lambda(\mathbf{a}) = \{\mathbf{p} = (p_1,p_2);\ p_i/d(a_i - 1) \geq \lambda \geq p_j/d(a_j)\}$. It may be shown that the probability in $X^\lambda(\mathbf{a})$ that one state is favored over the other is independent of the choice of λ.

A divisor method **M** is *unbiased for the pair* (a_1,a_2) if for some (equivalently, every) λ the probability in $X^\lambda(\mathbf{a})$ that state 1 is favored over state 2 equals the probability that state 2 is favored over state 1. **M** is *pairwise unbiased on apportionments* if **M** is unbiased for all pairs (a_1,a_2), $a_1 > a_2 > 0$.

THEOREM 8. *Webster's is the unique divisor method that is pairwise unbiased on apportionments.*

Another possible choice of sample space would be to let $Z(\mathbf{a}) = \{\mathbf{p} = (p_1,p_2);\ p_1 + p_2 = 1 \text{ and } p_i/d(a_i - 1) \geq p_j/d(a_j)\}$ for each pair $\mathbf{a} = (a_1,a_2)$, $a_1 > a_2 > 0$. With this model, *Theorem 8* also holds.

For arbitrary $s \geq 2$ and any apportionment $\mathbf{a} = (a_1, \ldots, a_s)$ and $\lambda > 0$, define $X^\lambda(\mathbf{a}) \subseteq \mathbf{R}^s_+$ as above. The interpretation of the pairwise approach in the context of s states is this. Choose any pair $a_1 > a_2 > 0$ and any apportionment $\mathbf{a} = (a_1,a_2, \ldots, a_s)$ having these first two coordinates. Then the probability that state 1 is favored over state 2 is the same as in the two-state model $X^\lambda(a_1,a_2)$ independently of λ and $a_3, \ldots, a_s$. Thus, the method of Webster is unbiased in the pairwise sense for any particular pair of states in an s-state problem.

An empirical test based on the pairwise approach is the following. For any empirical sample $\mathcal{S} = \{(\mathbf{p},\mathbf{a});\ \mathbf{p}$ a population vector, $\mathbf{a}$ an **M**-apportionment$\}$ let $\gamma(\mathcal{S})$ be the number of pairs of states (i,j) with $a_i > a_j > 0$ such that the large state is favored, $a_i/p_i > a_j/p_j$. Similarly, let $\delta(\mathcal{S})$ be the number of pairs of states in which the small state is favored. The *bias ratio* of the sample is defined to be $\delta(\mathcal{S})/[\gamma(\mathcal{S}) + \delta(\mathcal{S})]$. This test implicitly ignores any correlations that may exist between pairs chosen from the same s-state problem.

The five historic divisor methods have been applied to each of the 19 United States apportionment problems (1790–1970, inclusive) and the numbers δ and γ were counted (population data from ref. 19). To compare the overall tendencies, for each method the number of times the smaller state is favored over all 19 problems, $\Sigma_{\mathcal{S}}\delta(\mathcal{S})$, divided by the total number of comparisons, $\Sigma_{\mathcal{S}}[\gamma(\mathcal{S}) + \delta(\mathcal{S})]$, was computed to obtain the bias ratio over the course of Congressional apportionment. The results are given in Table 1.

Another test of bias is this. Take an apportionment $\mathbf{a} = (a_1, \ldots, a_s) > 0$ and let m (a *median*) be any number satisfying $a_i > m$ for at least $[s/2]$ states and $a_i < m$ for at least $[s/2]$ states. Define the *large states* to be the set $L = \{i;\ a_i > m\}$ and the *small states* the set $S = \{i; a_i < m\}$. A state with m seats belongs to both L and S. For $\mathbf{a}$ an **M**-apportionment for $\mathbf{p}$, the *large states are favored* if $\Sigma_L a_i/\Sigma_L p_i > \Sigma_S a_i/\Sigma_S p_i$ and the *small states are favored* if $\Sigma_S a_i/\Sigma_S p_i > \Sigma_L a_i/\Sigma_L p_i$. By using the sample spaces $X^\lambda(\mathbf{a})$ it may be shown that, for any divisor method **M**, the probability that the large states are favored is independent of the choice of $\lambda > 0$.

A divisor method **M** is *unbiased* if for every $\mathbf{a} > 0$ and $\lambda > 0$ the probability that the large states are favored in $X^\lambda(\mathbf{a})$ equals the probability that the small states are favored.

Table 1. Bias ratio over 1790–1970 U.S. Congressional apportionments

Method	Ratio
J. Q. Adams	0.780
J. Dean	0.583
E. V. Huntington	0.562
D. Webster	0.518
T. Jefferson	0.199

‖ It was fortunate, for this logic, that the number of methods considered was odd.

Table 2. Number of times small states were favored in 19 U.S apportionments

Method	Times, no.
J. Q. Adams	19
J. Dean	14
E. V. Huntington	13
D. Webster	9
T. Jefferson	0

THEOREM 9. *Webster's is the unique divisor method that is unbiased.*

The number of times the small states were favored by each of the five historic divisor methods over the course of U.S. Congressional apportionments is shown in Table 2. The method of Webster is clearly less biased than that used since 1940, Huntington's.

Minimum requirements

The U.S. Constitution requires that each state receive a minimum of 1 seat; France ensures each of its departments at least 2 seats; and the European Parliament has fixed minimum numbers of seats attached to each of the countries, ranging between 6 and 36. None of the preceding has explicitly accounted for a minimum requirement other than zero. However, with appropriate modifications of definitions, the theorems can be extended and the fundamental conclusions are the same.

Conclusion

Methods of apportionment must be analyzed by identifying the criteria they satisfy (or do not satisfy) and by observing their behavior when used for actual problems.

The argument of this paper may be summarized as follows. Population monotonicity for fixed s (=50) and h (=435) means that a divisor method must be used. Adjoining uniformity narrows the choice down to a divisor method defined independently of s and h and guarantees house monotonicity. The requirement, in addition, that a method not be biased toward small or large states leaves but one method: that of Webster.

The major loss appears to be the guarantee that apportionments satisfy quota. Insisting upon that guarantee would rule out all population monotone methods and all uniform methods. That is much too great a price. In fact, the method of Webster does "best" among the divisor methods in satisfying quota, and for three reasons.

First, as we have seen, it satisfies quota for $s = 3$, and is the only divisor method which does. Second, we say a method **M** is *near-quota* if there is no pair of states for which a transfer of one seat could simultaneously bring both of the states' apportionments nearer to their respective quotas.** The method of Webster is characterized as the unique divisor method which is near-quota (20). Third, empirical observation makes clear that the event of a Webster apportionment not satisfying quota is extremely unlikely. A Monte Carlo experiment confirms this: for $s = 50$, $h = 435$, 20,000 populations were chosen uniformly over the simplex $\{p;\ \Sigma p_i = 1,\ 435p_i \geq 0.5\}$. The method of Webster violated quota 37 times. This extrapolates to less than one violation of quota in 5000 years.

We conclude with Daniel Webster, ". . . let the rule be, that the population shall be divided by a common divisor, and, in addition to the number of members resulting from such division, a member shall be allowed to each state whose fraction exceeds a moiety of the divisor." (ref. 6, p. 120).

** This concept is equivalent to what we (20) have previously called "relatively well-rounded" and what Birkhoff (21) called "binary consistency."

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